# Exam Complex Analysis, 28 January 2014

The exam consists of 6 problems. Please write clearly and give a clear explanation of your answers. The maximal amount of points for each problem can be found below.

**1.** Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic on the domain $D$. Assume that $f(z)$ maps $D$ onto a portion of a line in the $w$-plane, i.e., there exist real numbers $a$, $b$ and $c$, with $a$ and $b$ not both equal to 0, such that $au(x, y) + bv(x, y) = c$ for all $z = x + iy \in D$.

**a.** Show that $a\frac{\partial u}{\partial x}(x, y) + b\frac{\partial v}{\partial x}(x, y) = 0$ and $a\frac{\partial u}{\partial y}(x, y) + b\frac{\partial v}{\partial y}(x, y) = 0$ on $D$.

**b.** Show that the partial derivatives of $u(x, y)$ and $v(x, y)$ are 0 on $D$.

**c.** Show that $f(z)$ is constant on $D$.

**2.** Consider the function $f(z) = \sin z$ on $\mathbb{C}$.

**a.** Show that the zeros of $f$ are real, and determine all zeros.

**b.** Write $f(z)$ in the form $u(x, y) + iv(x, y)$.

**c.** Use the Cauchy-Riemann equations to prove that $f(z)$ is an entire function.

**d.** Is $f(z)$ bounded on $\mathbb{C}$? Explain your answer.

**3.** Let $f(z)$ be analytic on and inside the simple closed contour $\Gamma$.

**a.** Let $z_0 \in \mathbb{C}$ be a point that does not lie inside or on $\Gamma$. Determine the integral

$$\int_\Gamma \frac{f(z)}{z - z_0} dz.$$

**b.** Let $g(z)$ be analytic on and inside $\Gamma$, such that $f(z) = g(z)$ for all $z$ on $\Gamma$. Prove that then $f(z) = g(z)$ for all $z$ inside $\Gamma$.

4. Consider the function $g(z) = \dfrac{e^{-z}}{(z+1)^2}$.

a. Find the Laurent series of $g(z)$ in $|z+1| > 0$.

b. Classify the singularity of $g(z)$

c. Let $\Gamma$ be the circle $|z| = 2$ traversed once in positive sense. Compute $\int_\Gamma g(z)dz$.

5. Consider the functie $f(z)$ given by $f(z) = z\cos(\dfrac{1}{2z})$.

a. Find the Laurent series of $f(z)$ in $|z| > 0$.

b. Classify the singularity of $f(z)$

d. Compute the residue of $f(z)$ in its singularity

6. Rouché's theorem is a very powerful result to determine information about the location of zeros of analytic functions.

a. Give a precise formulation of Rouché's theorem.

b. Determine the number of roots of the equation $6z^4+z^3-2z^2+z-\frac{7}{4}$ in the disc $|z| < 1$

c. Show that all roots lie in the annulus $\dfrac{1}{2} \leq |z| < 1$.

**Points:**

Problem 1: 16 (4 + 8 + 4)
Problem 2: 16 (4 + 4 + 4 + 4)
Problem 3: 16 (8 + 8)
Problem 4: 16 (8 + 4 + 4)
Problem 5: 16 (8 + 4 + 4)
Problem 6: 16 (4 + 6 + 6)
10 points for free